

# On the spectrum of the operator which is a composition of integration and substitution

by

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**Abstract.** Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ . Let the operator  $V_\phi : f(x) \rightarrow \int_0^{\phi(x)} f(t)dt$  be defined on  $L_2[0, 1]$ . We prove that  $V_\phi$  has a finite number of non-zero eigenvalues if and only if  $\phi(0) > 0$  and  $\phi(1 - \varepsilon) = 1$  for some  $0 < \varepsilon < 1$ . Also, we show that the spectral trace of the operator  $V_\phi$  always equals 1. <sup>0</sup>

## 1. Introduction.

It is well known that the Volterra integration operator  $V : f(x) \rightarrow \int_0^x f(t)dt$  defined on  $L_2[0, 1]$  is quasinilpotent, that is,  $\sigma(V) = \{0\}$ . Let  $\phi \in C[0, 1]$  such that  $\phi(0) = 0$ . It was pointed out in [9] and [10] that an operator  $V_\phi$  defined by

$$V_\phi : f(x) \rightarrow \int_0^{\phi(x)} f(t)dt \quad (1.1)$$

is quasinilpotent on  $C[0, 1]$  whenever  $\phi(x) \leq x$  for all  $x \in [0, 1]$ .

Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a measurable function and let  $V_\phi : L_p[0, 1] \rightarrow L_p[0, 1]$  ( $1 \leq p < \infty$ ) be defined by (1.1). It was proved in [12] and [13] that  $V_\phi$  is quasinilpotent on  $L_p[0, 1]$  if and only if  $\phi(x) \leq x$  for almost all  $x \in [0, 1]$ . It was noted in [13] and proved in [15] that the spectral radius of  $V_{x^\alpha}$  (defined on  $L_p[0, 1]$  or  $C[0, 1]$ ) is  $1 - \alpha$  ( $0 < \alpha < 1$ ). The detailed investigation of the spectrum of the operator  $V_{x^\alpha}$  was done in [1], where it was shown that the point spectrum  $\sigma_p(V_{x^\alpha})$  of  $V_{x^\alpha}$  is simple and  $\sigma_p(V_{x^\alpha}) = \{(1 - \alpha)\alpha^{n-1}\}_{n=1}^\infty$ . The oscillation properties of the eigenfunctions of  $V_{x^\alpha}$  also were investigated in [1].

The aim of this paper is to prove the following theorem.

**Theorem 1.1.** *Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ , and  $V_\phi$  be defined on  $L_2[0, 1]$  by (1.1). Set also  $\sigma_p(V_\phi) \setminus \{0\} = \{\lambda_n\}_{n=1}^\omega$  ( $1 \leq \omega \leq \infty$ ). Then:*

- (1)  $\omega < \infty$  if and only if  $\phi(0) > 0$  and  $\phi(1 - \varepsilon) = 1$  for some  $0 < \varepsilon < 1$ ;
- (2)  $\lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = 1$ ;
- (3)  $\sum_{n=1}^\omega |\lambda_n|^{1+\varepsilon} < \infty$  for all  $\varepsilon > 0$ .

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The order of the material is as follows.

In section 2 we recall some classical results in the theory of trace-class operators, in the theory of Fredholm determinants and in the theory of entire functions. In section 3 we calculate the Fredholm determinant  $D_{V_\phi}(\lambda)$  of the operator  $V_\phi$ . In section 4 we estimate the order of growth of  $D_{V_\phi}(\lambda)$  and prove Theorem 1.1. It turns out that the matrix trace of the operator  $V_\phi$  is not defined, but the spectral trace of  $V_\phi$  does not depend on  $\phi$  and always equals 1. This contrasts with the fact that  $\sigma_p(V_x) = \{\emptyset\}$ . We find also the spectral(= matrix) traces of the  $V_\phi^2$  and  $V_\phi^3$ . In section 5 we assume that  $\phi : [0, 1] \rightarrow [0, 1]$  is a strictly increasing continuous function such that  $\text{card}\{x : \phi(x) = x\} < \infty$  and describe the spectrum of  $V_\phi$ . Then we consider  $V_\phi$  defined on the space  $L_p[0, 1]$ .

**2. Preliminaries.** Here we recall some facts about trace class operators, Fredholm determinants and entire functions.

**2.1.** Let  $K$  be a compact operator defined on an infinite dimensional Hilbert space  $\mathfrak{H}$ . Let  $s_n(K)$  ( $n \geq 1$ ) be the eigenvalues of  $KK^*$ . The operator  $K$  is said to be of class  $\mathbf{S}_p$  if  $\sum_{n=1}^{\infty} s_n(K)^p < \infty$ . The trace  $\text{tr}K$  of an operator  $K \in \mathbf{S}_1$  is defined as its *matrix trace*:  $\text{tr}K = \sum_{n=1}^{\infty} (Ke_n, e_n)$ , where  $\{e_n\}_{n=1}^{\infty}$  is some orthonormal basis. It is known that  $\text{tr}K$  does not depend on the choice of  $\{e_n\}_{n=1}^{\infty}$  and the series  $\sum_{n=1}^{\infty} (Ke_n, e_n)$  converges absolutely. The celebrated theorem of Lidskii (see [4]) says that the matrix trace of an operator  $K \in \mathbf{S}_1$  is equal to its *spectral trace*, which is defined as the sum of eigenvalues of  $K$  (counted with the algebraic multiplicity):

$$\text{tr}K = \sum_{n=1}^{\infty} (Ke_n, e_n) = \sum_{n=1}^{\omega} \lambda_n, \quad \omega \leq \infty. \quad (2.2)$$

Let  $K$  be an integral operator:  $(Kf)(x) = \int_0^1 k(x, t)f(t)dt$  on  $L_2[0, 1]$ . It is well known (see [4]) that if  $k(x, t)$  is a continuous function on  $[0, 1] \times [0, 1]$ , then  $K \in \mathbf{S}_1$  and  $\text{tr}K$  is given by the integral of its diagonal:

$$\text{tr}K = \int_0^1 k(t, t)dt. \quad (2.3)$$

**2.2.** Now let  $k(x, t)$  be a bounded function on  $[0, 1] \times [0, 1]$ . By definition, put

$$D_K(\lambda) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n \lambda^n, \quad (2.4)$$

where  $A_0 := 1$  and

$$A_n := \int_0^1 \dots \int_0^1 K(t_1, \dots, t_n) dt_1 \dots dt_n, \quad (2.5)$$

$$K(t_1, \dots, t_n) := \det \begin{pmatrix} k(t_1, t_1) & \dots & k(t_1, t_n) \\ \vdots & & \vdots \\ k(t_n, t_1) & \dots & k(t_n, t_n) \end{pmatrix}$$

for  $n \geq 1$ . The function  $D_K(\lambda)$  is called *the Fredholm determinant* of  $K$ . Recall (see [6, 8, 11]) that:

$$(i) \quad A_n = n! \int_0^1 \int_{t_1}^1 \int_{t_2}^1 \dots \int_{t_{n-1}}^1 K(t_1, \dots, t_n) dt_n \dots dt_1, \quad n \geq 1; \quad (2.6)$$

- (ii)  $D_K(\lambda)$  is an entire function of  $\lambda$  of the order  $\rho \leq 2$ ;
- (iii)  $D_K(\mu^*) = 0$  if and only if  $\lambda^* := 1/\mu^* \in \sigma_p(K)$  and the multiplicity of  $\mu^*$  as a root of the Fredholm determinant of  $K$  is equal to the algebraic multiplicity of the eigenvalue  $\lambda^*$ .

**2.3.** From Hadamard's theorem (Th 1, p.26, [7]) and Lindelöf's theorem (Th 3, p.33, [7]), we get the following

**Theorem 2.2.** *Let  $f(z)$  be an entire function of order  $\rho_f \leq 1$  and type  $\sigma_f < \infty$ . Let also  $\{a_n\}_{n=1}^\omega$  ( $\omega \leq \infty$ ) be all roots of  $f(z)$  and  $f(0) = 1$ . Then*

$$(i) \text{ if } \rho_f = 1, \sigma_f = 0 \text{ and } \sum_{n=1}^\omega \frac{1}{|a_n|} < \infty, \text{ then } \omega = \infty, f(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right)$$

$$\text{and } \sum_{n=1}^\infty \frac{1}{a_n} = -f'(0);$$

$$(ii) \text{ if } \rho_f < 1, \text{ then } f(z) = \prod_{n=1}^\omega \left(1 - \frac{z}{a_n}\right) \text{ and } \sum_{n=1}^\omega \frac{1}{a_n} = -f'(0);$$

$$(iii) \text{ if } \rho_f = 0, \text{ then } \sum_{n=1}^\omega \frac{1}{|a_n|^\varepsilon} < \infty \text{ for each } \varepsilon > 0;$$

$$(iv) \text{ if } \rho_f = 1, \sigma_f = 0 \text{ and } \sum_{n=1}^\infty \frac{1}{|a_n|} = \infty,$$

$$\text{then } f(z) = e^{az} \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) e^{z/a_n} \text{ and } \limsup_{r \rightarrow \infty} \left| a + \sum_{|a_n| < r} \frac{1}{a_n} \right| = 0.$$

$$\text{In particular, } \limsup_{r \rightarrow \infty} \left( \sum_{|a_n| < r} \frac{1}{a_n} \right) = -a = -f'(0).$$

$$(v) \sum_{n=1}^\omega \frac{1}{|a_n|^{1+\varepsilon}} < \infty \text{ for each } \varepsilon > 0.$$

**3. The Fredholm determinant of the operator  $V_\phi$ .** We begin with an auxiliary lemma.

**Lemma 3.3.** Let  $A = (a_{ij})_{i,j=1}^n$  be an  $n \times n$  matrix all of whose elements are 0 or 1 and  $a_{ij} = 1$  for  $1 \leq j \leq i \leq n$ . Then

$$\det A = \prod_{i=2}^n (1 - a_{i-1i}) = \begin{cases} 1, & a_{i-1i} = 0 \text{ for } 2 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* The proof is trivial.  $\square$

**Theorem 3.4.** Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ . Let also  $V_\phi$  be defined on  $L_2[0, 1]$  by (1.1). Then

$$D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_{\phi(t_1)}^1 \dots \int_{\phi(t_{n-1})}^1 dt_n \dots dt_1. \quad (3.7)$$

*Proof.* It is clear that  $(V_\phi f)(x) = \int_0^1 k(x, t) f(t) dt =: (Kf)(x)$ , where

$$k(x, t) = \chi(\phi(x) - t) = \begin{cases} 1, & \phi(x) \geq t; \\ 0, & \phi(x) < t; \end{cases}$$

Assume that  $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1$ . Then  $k(t_i, t_j) = 1$  for  $1 \leq j \leq i \leq n$  and the matrix  $(k(t_i, t_j))_{i,j=1}^n$  satisfies the assumptions of Lemma 3.3. Hence,  $K(t_1, \dots, t_n) = \prod_{i=2}^n (1 - k(t_{i-1}, t_i))$ . Further, using (2.4), (2.5), and (2.6) we get

$$A_n = n! \int_0^1 \int_{t_1}^1 \int_{t_2}^1 \dots \int_{t_{n-1}}^1 \prod_{i=2}^n (1 - k(t_{i-1}, t_i)) dt_n \dots dt_1 = n! \int_{\Omega_n} 1 dt_n \dots dt_1,$$

where

$$\begin{aligned} \Omega_n &:= \{(t_1, \dots, t_n) : 0 \leq t_1 \leq \dots \leq t_n \leq 1, k(t_1, t_2) = \dots = k(t_{n-1}, t_n) = 0\} \\ &= \{(t_1, \dots, t_n) : 0 \leq t_1 \leq \phi(t_1) \leq t_2 \leq \phi(t_2) \leq \dots \leq \phi(t_{n-1}) \leq t_n \leq 1\}. \end{aligned}$$

That is

$$A_n = n! \int_0^1 \int_{\phi(t_1)}^1 \dots \int_{\phi(t_{n-1})}^1 dt_n \dots dt_1, \quad n \geq 1.$$

This completes the proof.  $\square$

#### 4. The spectrum of the operator $V_\phi$ .

The following Proposition immediately follows from Theorem 3.4.

**Proposition 4.5.** Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ . Then  $\sigma_p(V_\phi) \cap \mathbb{R}_- = \{\emptyset\}$ .

**Lemma 4.6.** Suppose  $\phi : [0, 1] \rightarrow [0, 1]$  is a nondecreasing continuous function and  $\phi(x) > x$  for  $x \in (0, 1)$ ; then the following conditions are equivalent:

- (i)  $\phi(0) > 0$  and  $\phi(1 - \varepsilon) = 1$  for some  $0 < \varepsilon < 1$ ;
- (ii) there exists a unique  $N = N(\phi) \in \{2, 3, \dots\}$  such that  $\phi^N(x) := \phi(\phi(\dots\phi(x))) = 1$  for all  $x \in [0, 1]$  and  $\phi^{N-1}(x_0) \neq 1$  for some  $x_0 \in [0, 1]$ .

*Proof.* The proof is left to the reader.  $\square$

**Theorem 4.7.** Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ . Suppose also that  $\phi(0) > 0$ ,  $\phi(1 - \varepsilon) = 1$  for some  $0 < \varepsilon < 1$ , and  $N = N(\phi)$  is determined by Lemma 4.6 (ii). Then

(1)  $\sigma_p(V_\phi) \setminus \{0\}$  is a finite set. Moreover,  $\sigma_p(V_\phi) = \{0\} \cup (\lambda_1, \dots, \lambda_N)$ , where  $\lambda_n \neq 0$ ;

$$(2) \sum_{n=1}^N \lambda_n = 1.$$

*Proof.* It is easily shown that  $0 \in \sigma_p(V_\phi)$ . Using Theorem 3.4, we get  $D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} A_n \lambda^n$ , where  $A_n = (-1)^n \int_0^1 \int_{\phi(t_1)}^1 \dots \int_{\phi(t_{n-1})}^1 dt_n \dots dt_1$ . It is easily shown that  $\phi^{n-1}(t_1) \leq t_n \leq 1$ . Since  $\phi^n(x) = 1$  for  $n \geq N$ , it follows that  $A_n = 0$  for  $n \geq N + 1$ . Therefore  $D_{V_\phi}(\lambda)$  is a polynomial of degree  $N$  and (1) is proved. Further note that  $D_{V_\phi}(\lambda) = \prod_{n=1}^N (1 - \frac{\lambda}{a_n})$ . Thus

$$\sum_{n=1}^N \lambda_n = \sum_{n=1}^N \frac{1}{a_n} = -A_1 = 1. \quad \square$$

Let  $\alpha_i(x), \beta_i(x) \in C[0, 1]$  ( $1 \leq i \leq n$ ). By definition, put

$$\left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \dots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\} := \int_{\beta_1(x)}^{\alpha_1(x)} \int_{\beta_2(x_1)}^{\alpha_2(x_1)} \dots \int_{\beta_n(x_{n-1})}^{\alpha_n(x_{n-1})} dx_n \dots dx_1.$$

So  $\left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \dots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\}$  is a function of  $x$ . It is clear that

$$\begin{aligned} & \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \dots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \alpha_i \\ \beta_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \dots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\} + \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \dots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \gamma_i \\ \alpha_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \dots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \dots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \alpha_i \\ \beta_i \end{matrix} + \begin{matrix} \gamma_i \\ \alpha_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \dots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \dots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \gamma_i \\ \beta_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \dots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\}. \end{aligned} \quad (4.8)$$

The following lemmas are needed.

**Lemma 4.8.** Let  $0 < \varepsilon_1 < \varepsilon_2 < 1$  and

$$\psi(x) = \begin{cases} \psi_1(x), & x \in [0, \varepsilon_1]; \\ \psi_2(x), & x \in [\varepsilon_1, \varepsilon_2]; \\ \psi_3(x), & x \in [\varepsilon_2, 1]; \end{cases}$$

be a strictly increasing continuous function such that  $\psi(\varepsilon_1) = \varepsilon_1$  and  $\psi(\varepsilon_2) = \varepsilon_2$ . Let also  $a_0 = b_0 = c_0 = 1$  and  $a_k, b_k, c_k, d_k$  ( $k = 1, 2, \dots$ ) be  $k$ -multiple integrals defined by

$$\begin{aligned} a_k &:= \left\{ \begin{matrix} \varepsilon_1 & \psi_1 & & \psi_1 \\ 0 & 0 & \dots & 0 \end{matrix} \right\}, & b_k &:= \left\{ \begin{matrix} \varepsilon_2 & \psi_2 & & \psi_2 \\ \varepsilon_1 & \varepsilon_1 & \dots & \varepsilon_1 \end{matrix} \right\}, \\ c_k &:= \left\{ \begin{matrix} 1 & \psi_3 & & \psi_3 \\ \varepsilon_2 & \varepsilon_2 & \dots & \varepsilon_2 \end{matrix} \right\}, & d_k &:= \left\{ \begin{matrix} 1 & \psi & & \psi \\ 0 & 0 & \dots & 0 \end{matrix} \right\}. \end{aligned}$$

Then

$$d_n = \sum_{k=0}^n c_k \sum_{l=0}^{n-k} b_l a_{n-k-l}, \quad n = 1, 2, \dots$$

*Proof.* Using (4.8), we get

$$\begin{aligned} d_n &= \left\{ \begin{matrix} \varepsilon_1 & \varepsilon_2 & 1 & \psi \\ 0 & \varepsilon_1 & \varepsilon_2 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} = \left\{ \begin{matrix} \varepsilon_1 & \psi_1 & & \psi_1 \\ 0 & 0 & \dots & 0 \end{matrix} \right\} \\ &+ \left\{ \begin{matrix} \varepsilon_2 & \varepsilon_1 & \psi_2 & \psi \\ \varepsilon_1 & 0 & \varepsilon_1 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} + \left\{ \begin{matrix} 1 & \varepsilon_1 & \varepsilon_2 & \psi_3 \\ \varepsilon_2 & 0 & \varepsilon_1 & \varepsilon_2 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} \\ &=: K_n + L_n + M_n. \end{aligned}$$

By definition  $K_n = a_n$ . Further, again using (4.8), we get

$$\begin{aligned} L_n &= \left\{ \begin{matrix} \varepsilon_2 & \varepsilon_1 & \psi_1 & \psi_1 \\ \varepsilon_1 & 0 & 0 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \varepsilon_2 & \psi_2 & \varepsilon_1 & \psi_2 \\ \varepsilon_1 & \varepsilon_1 & 0 & \varepsilon_1 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} \\ &= b_1 a_{n-1} + \left\{ \begin{matrix} \varepsilon_2 & \psi_2 & \varepsilon_1 & \psi_1 \\ \varepsilon_1 & \varepsilon_1 & 0 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \varepsilon_2 & \psi_2 & \psi_2 & \varepsilon_1 \\ \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} \\ &= b_1 a_{n-1} + b_2 a_{n-2} + \left\{ \begin{matrix} \varepsilon_2 & \psi_2 & \psi_2 & \psi_2 \\ \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} = \dots = \sum_{k=1}^n b_k a_{n-k}, \end{aligned}$$

$$\begin{aligned} M_n &= \left\{ \begin{matrix} 1 & \varepsilon_1 & \psi_1 & \psi_1 \\ \varepsilon_2 & 0 & 0 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} 1 & \varepsilon_2 & \psi_2 & \psi \\ \varepsilon_2 & \varepsilon_1 & 0 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} \\ &+ \left\{ \begin{matrix} 1 & \psi_3 & \psi & \psi \\ \varepsilon_2 & \varepsilon_2 & 0 & 0 \end{matrix} \right\} \\ &= c_1 a_{n-1} + c_1 L_{n-1} + \left\{ \begin{matrix} 1 & \psi_3 & \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_2 & 0 & \varepsilon_1 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} \\ &= c_1 a_{n-1} + c_1 L_{n-1} + c_2 a_{n-2} + c_2 L_{n-2} \\ &+ \left\{ \begin{matrix} 1 & \psi_3 & \psi_3 & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & 0 \end{matrix} \right\} + \left\{ \begin{matrix} \psi & & & \psi \\ 0 & & & 0 \end{matrix} \right\} \\ &= \dots = \sum_{k=1}^n c_k a_{n-k} + \sum_{k=1}^{n-1} c_k L_{n-k} = \sum_{k=1}^n c_k a_{n-k} + \sum_{k=1}^n c_k \sum_{l=1}^{n-k} b_l a_{n-k-l}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} d_n &= K_n + L_n + M_n = c_0 a_n + \sum_{k=1}^n b_k a_{n-k} + \sum_{k=1}^n c_k a_{n-k} + \sum_{k=1}^n c_k \sum_{l=1}^{n-k} b_l a_{n-k-l} \\ &= \sum_{k=0}^n c_k \sum_{l=0}^{n-k} b_l a_{n-k-l}. \end{aligned}$$

□

**Lemma 4.9.** *Let  $0 < \varepsilon \leq 1/4$ ,  $\beta > 1$ , and*

$$\psi_{\varepsilon, \beta}(x) = \begin{cases} x, & x \in [0, \varepsilon]; \\ \varepsilon + (1 - 2\varepsilon)^{1-\beta}(x - \varepsilon)^\beta, & x \in [\varepsilon, 1 - \varepsilon]; \\ x, & x \in [1 - \varepsilon, 1]; \end{cases}$$

Then

$$\begin{aligned} (1) \quad d_n &:= \left\{ \begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} \psi_{\varepsilon, \beta} \\ 0 \end{matrix}, \dots, \begin{matrix} \psi_{\varepsilon, \beta} \\ 0 \end{matrix} \right\} = \frac{(2\varepsilon)^n}{n!} + \frac{(1 - 2\varepsilon)(2\varepsilon)^{n-1}}{(n-1)!} \\ &\quad + \sum_{l=2}^n \frac{(1 - 2\varepsilon)^l (2\varepsilon)^{n-l}}{(n-l)!(1 + \beta) \dots (1 + \beta + \dots + \beta^{l-1})}, \quad n = 1, 2, \dots; \end{aligned} \quad (4.9)$$

$$(2) \quad d_n < \text{const}(\varepsilon, \beta) \frac{(4\varepsilon)^n}{n!}, \quad n = 1, 2, \dots,$$

where  $\text{const}(\varepsilon, \beta)$  does not depend on  $n$ .

*Proof.* Substituting  $\psi_{\varepsilon, \beta}$  for  $\psi(x)$  in Lemma 4.8, we get (4.9). Indeed, it is easily proved that  $a_l = c_l = \frac{\varepsilon^l}{l!}$  ( $l = 0, 1, \dots, n$ ). By definition, put  $\tilde{b}_1(x) := (1 - 2\varepsilon)^{1-\beta}(x - \varepsilon)^\beta$ ,  $\psi_2(x) := \varepsilon + \tilde{b}_1(x)$ , and

$$\tilde{b}_l(x) := \underbrace{\left\{ \begin{matrix} \psi_2 \\ \varepsilon \end{matrix}, \dots, \begin{matrix} \psi_2 \\ \varepsilon \end{matrix} \right\}}_l, \quad l = 2, 3, \dots$$

Then  $\tilde{b}_{l+1}(x) = \int_{\varepsilon}^{\psi_2(x)} \tilde{b}_l(t) dt$ . It can easily be checked (by induction on  $l$ ) that

$$\tilde{b}_l(x) = \frac{(1 - 2\varepsilon)^{l-\beta-\dots-\beta^l} (x - \varepsilon)^{\beta+\beta^2+\dots+\beta^l}}{(1 + \beta) \dots (1 + \beta + \dots + \beta^{l-1})}, \quad l = 2, 3, \dots$$

Since  $b_l = \tilde{b}_l(1 - \varepsilon)$ , we see that

$$b_0 = 1, \quad b_1 = 1 - 2\varepsilon, \quad b_l = \frac{(1 - 2\varepsilon)^l}{(1 + \beta) \dots (1 + \beta + \dots + \beta^{l-1})}, \quad l = 2, 3, \dots \quad (4.10)$$

Using Lemma 4.8, we get

$$\begin{aligned}
d_n &= \sum_{k=0}^n c_k \sum_{l=0}^{n-k} b_l a_{n-k-l} = \sum_{l=0}^n b_l \sum_{k=0}^{n-l} c_k a_{n-k-l} = \sum_{l=0}^n b_l \sum_{k=0}^{n-l} \frac{\varepsilon^k}{k!} \frac{\varepsilon^{n-k-l}}{(n-k-l)!} \\
&= \sum_{l=0}^n b_l \frac{\varepsilon^{n-l}}{(n-l)!} \sum_{k=0}^{n-l} \frac{(n-l)!}{k!(n-l-k)!} = \sum_{l=0}^n b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} \quad n = 1, 2, \dots
\end{aligned} \tag{4.11}$$

Substituting (4.10) for  $b_l$  in (4.11) we get (4.9).

(2) Taking into account the inequality of arithmetic and geometric means, we obtain

$$(1 + \beta) \dots (1 + \beta + \dots \beta^{l-1}) \geq 2\beta^{1/2} 3\beta^{2/2} \dots l\beta^{(l-1)/2} = l!\beta^{(l-1)l/4}. \tag{4.12}$$

Hence,

$$b_l \leq \frac{(1 - 2\varepsilon)^l}{l!} \left( \frac{1}{\beta^{1/4}} \right)^{l^2-l} < \frac{(1 - 2\varepsilon)^l}{l!}.$$

Let  $N$  be a number such that  $\left( \frac{1}{\beta^{1/4}} \right)^{l^2-l} < \left( \frac{2\varepsilon}{1-2\varepsilon} \right)^l$  for  $l > N$  (for example,  $N = [4 \log_\beta (\frac{1}{2\varepsilon} - 1)] + 2$ ). Then  $b_l < \frac{(2\varepsilon)^l}{l!}$  for  $l > N$ . Using (4.11), we get for  $n > N$

$$\begin{aligned}
d_n &= \sum_{l=0}^N b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} + \sum_{l=N+1}^n b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} \\
&\leq \frac{(2\varepsilon)^n}{n!} \sum_{l=0}^N \frac{n!}{l!(n-l)!} \left( \frac{1-2\varepsilon}{2\varepsilon} \right)^l + \frac{(2\varepsilon)^n}{n!} \sum_{l=N+1}^n \frac{n!}{l!(n-l)!} \\
&\leq \frac{(2\varepsilon)^n}{n!} \left( \frac{1-2\varepsilon}{2\varepsilon} \right)^N \sum_{l=0}^N \frac{n!}{l!(n-l)!} + \frac{(2\varepsilon)^n}{n!} \sum_{l=0}^n \frac{n!}{l!(n-l)!} \\
&\leq \frac{(4\varepsilon)^n}{n!} \left( \left( \frac{1-2\varepsilon}{2\varepsilon} \right)^N + 1 \right).
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 4.10.** *Let  $\beta > 1$  and*

$$\psi_\beta(x) := \begin{cases} 2^{\beta-1}x^\beta, & x \in [0, 1/2]; \\ 2^{\beta-1}(x - 1/2)^\beta + 1/2, & x \in [1/2, 1]; \end{cases} =: \begin{cases} \psi_1(x), & x \in [0, 1/2]; \\ \psi_2(x), & x \in [1/2, 1]; \end{cases}.$$

Let also  $a_0 = b_0 = 1$  and  $a_k, b_k$ , and  $d_k$  ( $k = 1, 2, \dots$ ) be  $k$ -multiple integrals defined by

$$\begin{aligned}
a_k &:= \left\{ \frac{1}{2}, \frac{\psi_1}{0}, \dots, \frac{\psi_1}{0} \right\}, \quad b_k := \left\{ \frac{1}{1/2}, \frac{\psi_2}{1/2}, \dots, \frac{\psi_2}{1/2} \right\}, \\
d_k &:= \left\{ \frac{1}{0}, \frac{\psi_\beta}{0}, \dots, \frac{\psi_\beta}{0} \right\}.
\end{aligned}$$

Then

$$(1) \quad d_n = \sum_{l=0}^n b_l a_{n-l}, \quad n = 1, 2, \dots; \quad (4.13)$$

$$(2) \quad d_n < \frac{\beta^{\frac{-n^2/2+n}{4}}}{n!}, \quad n = 1, 2, \dots$$

*Proof.* Substituting  $1/2$  for  $\varepsilon_1$  and  $1$  for  $\varepsilon_2$  in Lemma 4.8, we get (4.13).

Further, it is not hard to prove that  $a_1 = b_1 = 1/2$  and

$a_l = b_l = 2^{-l} ((\beta + 1) \cdots (\beta^{l-1} + \cdots + 1))^{-1}$  for  $l \geq 2$ . Now, by (4.12),  $a_l \leq \frac{2^{-l}}{\beta^{(l-1)l/4}l!}$  and

$$\begin{aligned} d_n &\leq \sum_{l=0}^n \frac{2^{-l}}{\beta^{(l-1)l/4}l!} \frac{2^{-n+l}}{\beta^{(n-l-1)(n-l)/4}(n-l)!} \\ &= \frac{2^{-n}}{n!} \sum_{l=0}^n \frac{n!}{l!(n-l)!} \beta^{\frac{-2(l-n/2)^2-n^2/2+n}{4}} < \frac{\beta^{\frac{-n^2/2+n}{4}}}{n!}. \end{aligned}$$

□

**Proposition 4.11.** *Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function.*

(1) *If  $\phi(x) > x$  for  $x \in (0, 1)$  then the order of  $D_{V_\phi}(\lambda)$  does not exceed 1, and*

*if it equals 1,  $D_{V_\phi}(\lambda)$  is of minimal type;*

(2) *if for some  $0 < a < b < 1$*

$$\phi(x) \geq f_{a,b}(x) := \begin{cases} \frac{b}{a}x, & x \in [0, a], \\ \frac{1-b}{1-a}x + \frac{b-a}{1-a}, & x \in [a, b], \end{cases}$$

*for  $x \in [0, 1]$ , then the order of  $D_{V_\phi}(\lambda)$  equals 0.*

*Proof.* (1) Taking into account Theorem 3.4, we obtain  $D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n A_n \lambda^n$ , where  $A_n = \left\{ \frac{1}{0}, \frac{1}{\phi}, \dots, \frac{1}{\phi} \right\}$ . Since  $\phi(x) > x$  for each  $0 < \varepsilon < 1/4$ , it follows that there exists  $\beta > 1$  such that  $\phi(x) \geq \psi_{\varepsilon, \beta}^{-1}(x)$ . Using Lemma 4.9, we get

$$\begin{aligned} A_n = d_n &= \left\{ \frac{1}{0}, \frac{1}{\phi}, \dots, \frac{1}{\phi} \right\} < \left\{ \frac{1}{0}, \frac{1}{\psi_{\varepsilon, \beta}^{-1}}, \dots, \frac{1}{\psi_{\varepsilon, \beta}^{-1}} \right\} \\ &= \left\{ \frac{1}{0}, \frac{\psi_{\varepsilon, \beta}}{0}, \dots, \frac{\psi_{\varepsilon, \beta}}{0} \right\} < \text{const}(\varepsilon, \beta) \frac{(4\varepsilon)^n}{n!}. \end{aligned}$$

Therefore the order of growth of  $D_{V_\phi}(\lambda)$  does not exceed 1. Assume that the order of growth of  $D_{V_\phi}(\lambda)$  is equal to 1. Then the type of  $D_{V_\phi}(\lambda)$  does not exceed  $4\varepsilon$  for each  $\varepsilon < 1/4$ . Thus  $D_{V_\phi}(\lambda)$  is of minimal type.

(2) Since  $\phi(x) \geq f_{a,b}(x)$  for some  $0 < a < b < 1$ , it follows that there exists  $\beta > 1$  such that  $\phi(x) \geq \psi_\beta^{-1}(x)$ . Using Lemma 4.10, we get

$$\begin{aligned} A_n = d_n &= \left\{ \begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & \phi & \cdots & \phi \end{matrix} \right\} < \left\{ \begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & \psi_\beta^{-1} & \cdots & \psi_\beta^{-1} \end{matrix} \right\} \\ &= \left\{ \begin{matrix} 1 & \psi_\beta & \cdots & \psi_\beta \\ 0 & 0 & \cdots & 0 \end{matrix} \right\} < \frac{\beta^{\frac{-n^2/2+n}{4}}}{n!}. \end{aligned}$$

Therefore the order of growth of  $D_{V_\phi}(\lambda)$  equals 0.  $\square$

**Theorem 4.12.** *Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ . Suppose that either  $\phi(0) = 0$  or  $\phi(1 - \varepsilon) \neq 1$  for all  $0 < \varepsilon < 1$ . Then*

- (1)  $\sigma_p(V_\phi) \setminus \{0\} := (\lambda_1, \dots, \lambda_n, \dots)$  — is an infinite set;
- (2)  $\lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = 1$ ;
- (3)  $\sum_{n=1}^{\omega} |\lambda_n|^{1+\varepsilon} < \infty$  for all  $\varepsilon > 0$ .

*Proof.* Using Theorem 3.4, we get  $D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n A_n \lambda^n$ , where  $A_n = \left\{ \begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & \phi & \cdots & \phi \end{matrix} \right\}$ . It is easy to see that if either  $\phi(0) = 0$  or  $\phi(1 - \varepsilon) \neq 1$  for all  $0 < \varepsilon < 1$ , then  $A_n > 0$  for  $n \geq 0$ . Therefore  $D_{V_\phi}(\lambda)$  is not a polynomial in  $\lambda$ . Now we apply Proposition 4.11 (1). Suppose that the order of  $D_{V_\phi}(\lambda)$  is less than 1; then using Theorem 2.2 (ii), we get  $D_{V_\phi}(\lambda) = \prod_{n=1}^{\omega} (1 - \frac{\lambda}{a_n})$ . Since  $D_{V_\phi}(\lambda)$  is not a polynomial, it follows that  $\omega = \infty$  and  $\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \frac{1}{a_n} = -A_1/A_0 = 1$ . Now suppose that the order of  $D_{V_\phi}(\lambda)$  is equal to 1; then  $D_{V_\phi}(\lambda)$  is of minimal type. Thus the spectrum of  $V_\phi$  is an infinite set. Now, the application of Theorem 2.2 (i), (iv) yields (2).

(3) follows from Theorem 2.2.  $\square$

Now we are ready to prove the main result of the paper

*Proof of Theorem 1.1*

(1) follows from Theorem 4.7 (1) and Theorem 4.12 (1).

(2)-(3) follow from Theorem 4.7 (2) and Theorem 4.12 (2)-(3).  $\square$

**Theorem 4.13.** *Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function and for some  $0 < a < b < 1$*

$$\phi(x) \geq \begin{cases} \frac{b}{a}x, & x \in [0, a], \\ \frac{1-b}{1-a}x + \frac{b-a}{1-a}, & x \in [a, b], \end{cases}$$

for all  $x \in [0, 1]$ . Let also either  $\phi(0) = 0$  or  $\phi(1 - \varepsilon) \neq 1$  for all  $0 < \varepsilon < 1$ . Then

- (1)  $\sigma_p(V_\phi) \setminus \{0\} := (\lambda_1, \dots, \lambda_n, \dots)$  — is an infinite set;
- (2)  $\sum_{n=1}^{\infty} \lambda_n = 1$ ;
- (3)  $\sum_{n=1}^{\infty} |\lambda_n|^\varepsilon < \infty$  for all  $\varepsilon > 0$ .

*Proof.* (1) follows from Theorem 4.12 (1). By Proposition 4.11 (2), the order of  $D_{V_\phi}(\lambda)$  equals 0. Thus (2) and (3) are implied by (ii) and (iii) of Theorem 2.2.  $\square$

**Remark 4.14.** (i) Suppose  $\phi(x)$  is a strictly increasing function and  $\phi(x) > x$  for all  $x \in (0, 1)$ . Let also  $\phi(x) \in C^1[0, 1]$  and  $(\phi'(x))^{-1/2} \in L_\infty[0, 1]$ . We claim that  $V_\phi \notin \mathbf{S}_1$ . Indeed, let  $c := \left( \int_0^1 (\phi'(s))^{1/2} ds \right)^{-1}$  and let  $W_\phi$  and  $T_\phi$  be linear operators defined on  $L_2[0, 1]$  by

$$(W_\phi f)(x) = \int_0^x (\phi'(t))^{1/2} f(t) dt, \quad (T_\phi f)(x) = f\left(c \int_0^x (\phi'(s))^{1/2} ds\right).$$

It can easily be checked (see [2]-[3]) that  $T_\phi$  and  $T_\phi^{-1}$  are bounded operators and  $cV_x = T_\phi^{-1}W_\phi T_\phi$ . Hence, (see [5])  $s_n(W_\phi) \geq \|T_\phi\|^{-1} \|T_\phi^{-1}\|^{-1} s_n(cV_x) = \|T_\phi\|^{-1} \|T_\phi^{-1}\|^{-1} c \frac{2}{(2n-1)\pi}$ . Further,

$$(V_\phi V_\phi^* f)(x) = \int_0^{\phi(x)} \int_{\phi^{-1}(t)}^1 f(s) ds dt = \int_0^x \phi'(t) \int_t^1 f(s) ds dt = (W_\phi W_\phi^* f)(x).$$

Thus  $s_n(V_\phi) = s_n(W_\phi) \geq \|T_\phi\|^{-1} \|T_\phi^{-1}\|^{-1} c \frac{2}{(2n-1)\pi}$ . Hence,  $V_\phi \notin \mathbf{S}_1$ .

(ii) Since  $V_\phi \notin \mathbf{S}_1$ , it follows that the matrix trace of an operator  $V_\phi$  is not defined. Hence we cannot use (2.2)-(2.3) to prove Theorem 4.13 (2). Nevertheless, (2.2)-(2.3) hold for  $K = V_\phi$  and the orthonormal basis  $\{e_n\}_{n=1}^\infty$  defined by:  $e_1 \equiv 1$ ,  $e_{2n} := e^{2\pi i n x}$  and  $e_{2n+1} := e^{-2\pi i n x}$  ( $n = 1, 2, \dots$ ). Indeed, since  $\sum_{n=1}^\infty \frac{\sin nx}{n} = \frac{\pi-x}{2}$  for  $x \in (0, 2\pi)$ , it follows that

$$\begin{aligned} \sum_{n=0}^\infty (V_\phi e_n, e_n) &= \int_0^1 \phi(x) dx \\ &+ \sum_{n=1}^\infty \left( \int_0^1 \frac{(e^{2\pi i n \phi(x)} - 1)e^{-2\pi i n x}}{2\pi i n} dx + \int_0^1 \frac{(e^{-2\pi i n \phi(x)} - 1)e^{2\pi i n x}}{-2\pi i n} dx \right) \\ &= \int_0^1 \phi(x) dx + \sum_{n=1}^\infty \int_0^1 \frac{\sin(2\pi n(\phi(x) - x))}{\pi n} dx \\ &= \int_0^1 \phi(x) dx + \int_0^1 \frac{1}{\pi} \frac{(\pi - 2\pi(\phi(x) - x))}{2} dx = 1. \end{aligned}$$

Further,  $\int_0^1 \chi(\phi(x) - x)dx = 1$ . Thus formulas (2.2)-(2.3) hold. This contrasts with the fact that  $\sum_{n=0}^{\infty} (V_x e_n, e_n) = \infty$ .

(iii) Theorem 1.1 states that the spectral trace of an operator  $V_\phi$  always equals 1. This also contrasts with the fact that an operator  $V_x$  is quasinilpotent.

To estimate the spectral radius  $r(V_\phi)$  of the operator  $V_\phi$  we recall (see [14]) some results on integral operators with nonnegative kernels. Let  $(Kf)(x) = \int_0^1 k(x, t)f(t)dt$  and  $k(x, t) \geq 0$  for  $(x, t) \in [0, 1] \times [0, 1]$ . If there exist  $\alpha > 0$  and a nonnegative function  $f$  such that  $(Kf)(x) \geq \alpha f(x)$  for  $x \in [0, 1]$ , then  $r(K) \in \sigma_p(K)$  and  $r(K) > \alpha$ .

**Proposition 4.15.** *Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a strictly increasing continuous function such that  $\phi(x) \geq x$  for all  $x \in [0, 1]$ . Set also  $\sigma_p(V_\phi) = \{\lambda_n\}_{n=1}^\omega$  ( $\omega \leq \infty$ ). Then*

$$(1) \ r(V_\phi) \geq \max_{x \in [0, 1]} (\phi(x) - x), \quad r(V_\phi) \in \sigma_p(V_\phi).$$

Let also  $\phi(0) = 0$ . Then  $\omega = \infty$  and

$$(2) \ \sum_{n=1}^{\infty} \lambda_n^2 = 2 \int_0^1 \phi(t)dt - 1;$$

$$(3) \ \sum_{n=1}^{\infty} \lambda_n^3 = 1 - 3 \int_0^1 \phi(t)\phi^{-1}(t)dt.$$

*Proof.* (1) Let  $f_a(x) = 1 - \chi(a - x)$ ,  $a \in (0, 1)$  then

$$(V_\phi f_a)(x) = \begin{cases} 0, & [0, \phi^{-1}(a)] \\ \phi(x) - a, & [\phi^{-1}(a), 1] \end{cases} \geq (\phi(a) - a)f_a(x),$$

and (1) is proved.

(2), (3) It is easy to check that  $\phi^{-1}(x)$  is well defined and

$$(V_\phi^2 f)(x) = \int_0^1 \chi(\phi^2(x) - t)(\phi(x) - \phi^{-1}(t))f(t)dt =: \int_0^1 k_2(x, t)f(t)dt.$$

$$(V_\phi^3 f)(x) = \int_0^1 \chi(\phi^3(x) - t) \int_{\phi^{-2}(t)}^{\phi(x)} (\phi(s) - \phi^{-1}(t))ds f(t)dt =: \int_0^1 k_3(x, t)f(t)dt.$$

Further,  $k_2(x, t)$  and  $k_3(x, t)$  are continuous functions on  $[0, 1] \times [0, 1]$ . Hence,  $V_\phi^2 \in \mathbf{S}_1$  and  $V_\phi^3 \in \mathbf{S}_1$ . Now if we recall (2.3), we get

$$\sum_{n=1}^{\infty} \lambda_n^2 = \int_0^1 k_2(t, t)dt = \int_0^1 (\phi(t) - \phi^{-1}(t))dt = 2 \int_0^1 \phi(t)dt - 1,$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n^3 &= \int_0^1 k_3(t, t) dt = \int_0^1 \int_{\phi^{-2}(t)}^{\phi(t)} (\phi(s) - \phi^{-1}(t)) ds \\
&= \int_0^1 (\phi(t)\phi^2(t) - 2\phi^{-1}(t)\phi(t) + \phi^{-1}(t)\phi^{-2}(t)) dt = 1 - 3 \int_0^1 \phi(t)\phi^{-1}(t) dt.
\end{aligned}$$

□

**Example 4.16.** Let  $\phi(x) = x^\alpha$  ( $0 < \alpha < 1$ ). It can be proved by direct calculations that

$$\begin{aligned}
D_{V_{x^\alpha}}(\lambda) &= 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_{t_1^\alpha}^1 \dots \int_{t_{n-1}^\alpha}^1 dt_n \dots dt_1 \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \frac{\alpha^{n(n-1)/2} (1-\alpha)^n}{(1-\alpha) \dots (1-\alpha^n)} = \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda}{(1-\alpha)\alpha^{n-1}} \right).
\end{aligned}$$

Hence,  $\sigma_p(V_{x^\alpha}) = \{(1-\alpha)\alpha^{n-1}\}_{n=1}^{\infty}$  and each eigenvalue of  $V_{x^\alpha}$  is of algebraic multiplicity one. Further,  $\sum_{n=1}^{\infty} (1-\alpha)\alpha^{n-1} = 1$  and  $\sum_{n=1}^{\infty} ((1-\alpha)\alpha^{n-1})^\varepsilon = \frac{(1-\alpha)^\varepsilon}{1-\alpha^\varepsilon} < \infty$  for each  $\varepsilon > 0$ .

## 5. Some generalizations.

**5.1.** The following Lemma can be proved by direct calculations.

**Lemma 5.17.** Let  $A$  be a compact operator defined on a Hilbert space  $\mathfrak{H}$ . Let also  $\mathfrak{H} = \bigoplus_{i=1}^k \mathfrak{H}_i$  and  $A_i := P_i A : \mathfrak{H}_i \rightarrow \mathfrak{H}_i$ , where  $P_i$  be an orthoprojection in  $\mathfrak{H}$  onto  $\mathfrak{H}_i$ . Suppose that  $\{\bigoplus_{j=1}^i \mathfrak{H}_j\}_{i=1}^k$  is invariant for  $A$ ; then  $1/\lambda$  is an eigenvalue of  $A$  of the algebraic multiplicity  $m \geq 1$  if and only if  $1/\lambda$  is an eigenvalue of  $A_i$  of the algebraic multiplicity  $m_i \geq 0$  and  $\sum_{i=1}^k m_i = m$ .

*Proof.* The proof is omitted. □

**Theorem 5.18.** Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a strictly increasing continuous function. Let also  $\{x : \phi(x) = x, x \in (0, 1)\} = \{a_i\}_{i=1}^{k-1}$ , where  $0 < a_1 < \dots < a_{k-1} < 1$  ( $k \geq 2$ ). By definition, put  $a_0 := 0$ ,  $a_k := 1$ , and

$$\phi_i(x) := (\phi(x(a_i - a_{i-1}) + a_{i-1}) - a_{i-1}) / (a_i - a_{i-1}), \quad 1 \leq i \leq k.$$

$$D_{V_{\phi_i}}(\lambda) := \begin{cases} 1 + \sum_{n=0}^{\infty} (-\lambda)^n \left\{ \frac{1}{0}, \frac{1}{\phi_i}, \dots, \frac{1}{\phi_i} \right\}, & \phi_i(x) > x \text{ for } x \in (0, 1); \\ 1, & \phi_i(x) < x \text{ for } x \in (0, 1). \end{cases}$$

Then

- (1)  $1/\lambda \in \sigma_p(V_\phi)$  if and only if  $\prod_{i=1}^k D_{V_{\phi_i}}((a_i - a_{i-1})\lambda) = 0$ ;  
(2) the algebraic multiplicity of the eigenvalue  $1/\lambda$  is equal to the multiplicity of  $\lambda$  as a root of the entire function  $\prod_{i=1}^k D_{V_{\phi_i}}((a_i - a_{i-1})\lambda)$ .

*Proof.* By definition, put  $\mathfrak{H} := L_2[0, 1]$ ,  $\mathfrak{H}_i := L_2[a_{i-1}, a_i]$  and

$$\begin{aligned} P_i : f(x) &\rightarrow \begin{cases} f(x), & x \in [a_{i-1}, a_i]; \\ 0, & x \notin [a_{i-1}, a_i]; \end{cases}, & P_i : \mathfrak{H} &\rightarrow \mathfrak{H}_i, \\ A &:= V_\phi, & A_i &:= P_i A \upharpoonright_{\mathfrak{H}_i}, \\ T_i : \begin{cases} f(x), & x \in [a_{i-1}, a_i]; \\ 0, & x \notin [a_{i-1}, a_i]; \end{cases} &\rightarrow f((a_i - a_{i-1})x + a_{i-1}), & T_i : \mathfrak{H}_i &\rightarrow \mathfrak{H}. \end{aligned}$$

It follows easily that  $\bigoplus_{j=1}^i \mathfrak{H}_j (= L_2[0, a_i])$  is invariant for  $A$  and

$$\begin{aligned} A_i : \begin{cases} f(x), & x \in [a_{i-1}, a_i]; \\ 0, & x \notin [a_{i-1}, a_i]; \end{cases} &\rightarrow \begin{cases} \int_{a_{i-1}}^{\phi(x)} f(t) dt, & x \in [a_{i-1}, a_i]; \\ 0, & x \notin [a_{i-1}, a_i]; \end{cases}, \\ T_i^{-1} : f(x) &\rightarrow \begin{cases} f(\frac{x-a_{i-1}}{a_i-a_{i-1}}), & x \in [a_{i-1}, a_i]; \\ 0, & x \notin [a_{i-1}, a_i]; \end{cases}, & T_i : \mathfrak{H} &\rightarrow \mathfrak{H}_i, \\ T_i A_i T_i^{-1} &= (a_i - a_{i-1}) V_{\phi_i}. \end{aligned}$$

The application of Theorem 3.4 yields

$$1/\lambda \in \sigma_p(A_i) \Leftrightarrow 1/\lambda \in \sigma_p((a_i - a_{i-1})V_{\phi_i}) \Leftrightarrow D_{V_{\phi_i}}((a_i - a_{i-1})\lambda) = 0.$$

The applying of Lemma 5.17 completes the proof.  $\square$

**Corollary 5.19.** Suppose  $\phi(x)$  satisfies the conditions of Theorem 5.18 and  $\text{mes}\{x : \phi(x) \geq x, x \in [0, 1]\} > 0$ . Set also  $\sigma_p(V_\phi) \setminus \{0\} = \{\lambda_n\}_{n=1}^\omega$  ( $1 \leq \omega \leq \infty$ ). Then

(1)  $\omega < \infty$  if and only if  $\phi(0) > 0$ ,  $\phi(1 - \varepsilon) = 1$  for some  $0 < \varepsilon < 1$  and  $\phi(x) > x$  for all  $x \in (0, 1)$ ;

$$(2) \quad \lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = \text{mes}\{x : \phi(x) \geq x, x \in [0, 1]\}.$$

*Proof.* (1) follows from Theorems 4.7, 4.12, 5.18.

(2) By definition, put

$$\begin{aligned} \Omega &:= \{i : \phi(x) \geq x \text{ for } x \in [a_{i-1}, a_i]\} = \{i : \phi_i(x) \geq x \text{ for } x \in [0, 1]\}, \\ \sigma_p(V_{\phi_i}) &:= \{\lambda_{in}\}_{n=1}^{\omega_i}, \quad 1 \leq \omega \leq \infty, \quad i \in \Omega. \end{aligned}$$

By Theorem 5.18

$$\{\lambda_n\}_{n=1}^\omega = \sigma_p(V_\phi) = \bigcup_{i \in \Omega} \sigma_p((a_i - a_{i-1})V_{\phi_i}) = \bigcup_{i \in \Omega} (a_i - a_{i-1})\{\lambda_{in}\}_{n=1}^{\omega_i}.$$

By Theorem 4.12

$$\lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_{in}| > \varepsilon} \lambda_{in} = 1.$$

Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n &= \sum_{i \in \Omega} (a_i - a_{i-1}) \lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_{in}| > \varepsilon} \lambda_{in} \\ &= \sum_{i \in \Omega} (a_i - a_{i-1}) = \text{mes}\{x : \phi(x) \geq x, x \in [0, 1]\}. \end{aligned}$$

□

**Remark 5.20.** *It is interesting to note that the case of nonincreasing function  $\phi$  can be more multifarious. In particular, if  $\phi(x)$  is a strictly decreasing continuous function such that  $\phi(0) = 1$ ,  $\phi(1) = 0$  and  $\phi(\phi(x)) = x$  then  $V_\phi$  is a selfadjoint operator in  $L_2[0, 1]$ . For example,  $\sigma_p(V_{1-x}) = \{\frac{2(-1)^n}{(2n+1)\pi}\}_{n=1}^\infty$  and  $\sum_{n=1}^\infty \frac{2(-1)^n}{(2n+1)\pi} = \frac{2}{\pi} \frac{\pi}{4} = \frac{1}{2} = \text{mes}\{x : 1 - x \geq x\}$ .*

**5.2.** In this subsection we consider an operator  $V_\phi$  defined on  $L_p[0, 1]$  ( $1 \leq p < \infty$ ).

Let  $A_i$  be a bounded operator defined on Banach space  $X_i$  ( $i = 1, 2$ ). Recall that  $A_1$  is said to be quasisimilar to  $A_2$  if there exist deformations  $K : X_1 \rightarrow X_2$  and  $L : X_2 \rightarrow X_1$  (i.e.  $\overline{\mathfrak{R}(K)} = X_2$ ,  $\ker K = \{0\}$ ,  $\overline{\mathfrak{R}(L)} = X_1$ ,  $\ker L = \{0\}$ ) such that  $A_1 L = L A_2$  and  $K A_1 = A_2 K$ . It is clear that  $\sigma_p(A_1) = \sigma_p(A_2)$ .

**Proposition 5.21.** *Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a strictly increasing continuous function such that  $\phi(0) = 0$  and  $\phi(1) = 1$ . Let  $A_1$  denote an operator  $V_\phi$  defined on  $L_p[0, 1]$  ( $1 \leq p < \infty$ ) and let  $A_2$  denote an operator  $V_\phi$  defined on  $L_2[0, 1]$ . Then  $A_1$  is quasisimilar to  $A_2$ , and hence  $\sigma_p(A_1) = \sigma_p(A_2)$ .*

*Proof.* By definition, put  $K := V_\phi : L_p[0, 1] \rightarrow L_2[0, 1]$ ,  $L := V_\phi : L_2[0, 1] \rightarrow L_p[0, 1]$ . It is clear that  $K$  and  $L$  are deformations and  $A_1 L = L A_2$ ,  $K A_1 = A_2 K$ . □

**5.3.** Now we consider the operator  $(V_{\phi, q, w} f)(x) := q(x) \int_0^{\phi(x)} f(t) w(t) dt$  defined on  $L_2[0, 1]$ .

The proof of the following theorem is similar to the proof of Theorem 3.4.

**Theorem 5.22.** Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a nondecreasing continuous function such that  $\phi(x) > x$  for all  $x \in (0, 1)$ . Let also  $q(x), w(x) \in L_2[0, 1]$ . Then

$$D_{V_{\phi,q,w}}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_{\phi(t_1)}^1 \dots \int_{\phi(t_{n-1})}^1 q(t_1)w(t_1) \dots q(t_n)w(t_n) dt_n \dots dt_1.$$

**Corollary 5.23.** Let the conditions of Theorem 5.22 hold and  $q(x)w(x) > 0$  for a.a.  $x \in [0, 1]$ . Then  $\sigma_p(V_{\phi,q,w}) \setminus \{0\}$  is a finite set if and only if  $\phi(0) > 0$  and  $\phi(1 - \varepsilon) = 1$  for some  $0 < \varepsilon < 1$ .

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